ON THE GENERATORS OF QUANTUM STOCHASTIC OPERATOR COCYCLES

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Dedicated to the memory of John Lewis — an inspirational academic great-grandfather.

ABSTRACT. The stochastic generators of Markov-regular operator cocycles on symmetric Fock space are studied in a variety of cases: positive cocycles, projection cocycles, and partially isometric cocycles. Moreover a class of transformations of positive contraction cocycles is exhibited which leads to a polar decomposition result.

0. INTRODUCTION

Let H be a Hilbert space and $\sigma = (\sigma_t)_{t\geqslant 0}$ be a semigroup of unital endomorphisms of the algebra $B(\mathsf{H})$ — an E_0-semigroup ([Arv03]). A family of operators $X = (X_t)_{t\geqslant 0} \subset B(\mathsf{H})$ is a *left cocycle* (respectively a *right cocycle*) for σ if it satisfies

$$X_0 = I \quad \text{and} \quad X_{r+t} = X_r \sigma_r(X_t) \quad (\text{resp. } X_{r+t} = \sigma_r(X_t)X_r) \tag{0.1}$$

for all $r, t \geqslant 0$. *Unitary* cocycles (i.e. each X_t is unitary) play a fundamental role in the classification of E_0-semigroups, which is carried out up to conjugation by such objects. The E_0-semigroups of type I (those that possess a sufficiently large number of cocycles) turn out to be precisely those that are cocycle conjugate to the CCR flow on symmetric Fock space, the particular flow being uniquely specified by a choice of Hilbert space k, called the *noise dimension space* in the language of quantum stochastic calculus (QSC). In QSC cocycles arise naturally as solutions of a quantum stochastic differential equation (QSDE) of Hudson-Parthasarathy type, and it is standard practice to ampliate the CCR flow so that it acts on $B(\mathfrak{h} \otimes \mathcal{F}_+)$, where $\mathfrak{h}$ is another Hilbert space (the *initial space*) and $\mathcal{F}_+$ denotes the symmetric Fock space over $L^2(\mathbb{R}_+; \mathsf{k})$. The coefficient driving the QSDE is some operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, where $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, and (conjugation by) the resulting cocycle can be viewed as a Feynman-Kac perturbation of the free evolution given by σ ([Acc78]). Conversely, any contraction cocycle that is *Markov-regular* necessarily satisfies such a QSDE for some such F, and moreover the collection of those F that are generators of contraction cocycles is now well-known, as are the subsets corresponding to the generators of isometric, coisometric and unitary cocycles. For more details see [Fag93, LP98, LW00a, LW00b], or the lecture notes [Lin05].

In this paper we characterise the generators of many other classes of cocycle, namely self-adjoint cocycles, positive cocycles, projection cocycles and partially isometric cocycles, going beyond the case of contractive cocycles for the first two classes. Positive contraction cocycles have appeared in the work of Bhat ([Bha01]), where they are used to study dilations and compressions between E-semigroups and quantum Markov semigroups on $B(\mathsf{H})$. In the third section of the paper we discuss a one-parameter family of transformations on the class of positive contraction cocycles, and describe the corresponding transformation on the stochastic generators

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in the Markov-regular case. This leads naturally to a polar decomposition result in the final section, where it is shown that any Markov-regular contraction cocycle with commutative component von Neumann algebra can be written as a product of a partial isometry cocycle and a positive cocycle.

Notational conventions. Algebraic tensor products are denoted by $\underline{\otimes}$, with $\otimes$ reserved for the (completed) tensor product of Hilbert spaces and the tensor product of von Neumann algebras. The tensor symbol between Hilbert space vectors in elementary tensors will usually be suppressed. Given Hilbert spaces H and h, and $x \in \mathsf{h}$, we define maps $E_x \in B(\mathsf{H}; \mathsf{H} \otimes \mathsf{h})$ and $E^x \in B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H})$ by

$$E_x = u \otimes x = ux \quad \text{and} \quad E^x = (E_x)^*,$$

with context indicating the choice of H and h.

1. Operator cocycles on Fock space

Fix two Hilbert spaces, the initial space $\mathfrak{h}$ and the noise dimension space k. Let $\mathcal{F}_+$ denote the symmetric Fock space over $L^2(\mathbb{R}_+; \mathsf{k})$, and in general let $\mathcal{F}_I$ denote the symmetric Fock space over $L^2(I; \mathsf{k})$ for $I \subset \mathbb{R}$. We shall make frequent use of the time shift and time reversal operators on $\mathfrak{h} \otimes \mathcal{F}_+$, S_t and R_t respectively, which are the ampliated second quantisations of

$$(s_t f)(u) = \begin{cases} 0 & \text{if } u < t, \\ f(u-t) & \text{if } u \geqslant t, \end{cases} \quad \text{and} \quad (r_t f)(u) = \begin{cases} f(t-u) & \text{if } u \leqslant t, \\ f(u) & \text{if } u > t, \end{cases}$$

so that $S_t u\varepsilon(f) = u\varepsilon(s_t f)$, where $\varepsilon(f) = (1, f, (2!)^{-1/2} f \otimes f, \ldots)$ is the exponential vector associated to $f \in L^2(\mathbb{R}_+; \mathsf{k})$. Note that the S_t are isometries and the R_t are self-adjoint unitaries, with both maps $t \mapsto S_t$ and $t \mapsto R_t$ continuous in the strong operator topology, that is, *strongly continuous*. The endomorphism semigroup $(\sigma_t)_{t \geqslant 0}$ on $B(\mathfrak{h} \otimes \mathcal{F}_+)$ is constructed from $(S_t)_{t \geqslant 0}$ by using the obvious isomorphism these maps induce between $\mathcal{F}_+$ and $\mathcal{F}_{[t,\infty[}$. More concretely, for any $X \in B(\mathfrak{h} \otimes \mathcal{F}_+)$ the operator $\sigma_t(X)$ is determined by

$$\langle u\varepsilon(f), \sigma_t(X) v\varepsilon(g) \rangle = \langle u\varepsilon(s_t^* f), X v\varepsilon(s_t^* g) \rangle \exp \int_0^t \langle f(u), g(u) \rangle \, du. \tag{1.1}$$

The time reversal operators on $B(\mathfrak{h} \otimes \mathcal{F}_+)$ are

$$\rho_t(X) = R_t X R_t. \tag{1.2}$$

A *Fock-adapted* left (respectively right) cocycle on $\mathfrak{h} \otimes \mathcal{F}_+$ is any family $X = (X_t)_{t \geqslant 0} \subset B(\mathfrak{h} \otimes \mathcal{F}_+)$ that satisfies the functional equation (0.1) together with the adaptedness condition

$$X_t \in B(\mathfrak{h} \otimes \mathcal{F}_{[0,t[}) \otimes I_{\mathcal{F}_{[t,\infty[}}, \tag{1.3}$$

where we utilise the continuous tensor product factorisation of Fock space: $\mathcal{F}_+ \cong \mathcal{F}_{[0,t[} \otimes \mathcal{F}_{[t,\infty[}$ via $\varepsilon(f) \longleftrightarrow \varepsilon(f|_{[0,t[}) \otimes \varepsilon(f|_{[t,\infty[})$. All cocycles in this paper will be assumed to satisfy (1.3).

Continuity was not given as part of the definition; the next result mirrors/relies on the corresponding result in semigroup theory.

Proposition 1.1. *Let $X = (X_t)_{t \geqslant 0}$ be a left cocycle on $\mathfrak{h} \otimes \mathcal{F}_+$. If $X_t \to I$ weakly as $t \to 0$ then there are constants $M, a \in \mathbb{R}$ such that*

$$\|X_t\| \leqslant M e^{at} \quad \text{for all } t \geqslant 0. \tag{1.4}$$

Moreover, the map $t \mapsto X_t$ is strongly continuous.

Cocycles satisfying these continuity conditions will be called *C_0-cocycles*.

Proof. Weak convergence to I implies, via two applications of the Banach-Steinhaus Theorem, that $t \mapsto \|X_t\|$ is bounded in a neighbourhood of 0. The existence of M and a then follows by a standard argument (see, for example, Proposition 1.18 of [Dav80]) since $\|X_{r+t}\| \leqslant \|X_r\| \|X_t\|$, because each σ_t is contractive.

Weak continuity at 0, and the local uniform bound for X from (1.4) imply that $X_t \otimes I_- \to I_{\mathfrak{h}\otimes\mathcal{F}_\mathbb{R}}$ weakly, where I_- is the identity on $\mathcal{F}_- := \mathcal{F}_{]-\infty,0[}$, and we use $\mathcal{F}_\mathbb{R} \cong \mathcal{F}_+ \otimes \mathcal{F}_-$. If $(\overline{S}_t)_{t\geqslant 0}$ denotes the strongly continuous family of *unitary* right shifts on $\mathfrak{h} \otimes \mathcal{F}_\mathbb{R}$ defined analogously to the isometries S_t, then $Y_t := (X_t \otimes I_-)\overline{S}_t$ is weakly convergent to $I_{\mathfrak{h}\otimes\mathcal{F}_\mathbb{R}}$. Moreover, for any $Z \in B(\mathfrak{h} \otimes \mathcal{F}_+)$

$$\sigma_t(Z) \otimes I_- = \overline{S}_t(Z \otimes I_-)\overline{S}_t^*,$$

and it readily follows that $(Y_t)_{t\geqslant 0}$ is a semigroup on $\mathfrak{h} \otimes \mathcal{F}_\mathbb{R}$. Hence it is strongly continuous, by Proposition 1.23 of [Dav80], thus so is $t \mapsto X_t \otimes I_- = Y_t\overline{S}_t^*$, and the result follows. □

Remarks. (i) A result in the same spirit is Proposition 2.5 of [Arv89] (reappearing as Proposition 2.3.1 of [Arv03]). It is more general on the one hand since it only assumes measurability of the cocycle, which is defined with respect to a *general* E_0-semigroup on a von Neumann algebra. However there are separability assumptions, and essential use is made of the more restrictive hypothesis that the cocycle be isometric. Similarly, assumed contractivity of the cocycle is a necessary ingredient of the alternative proof of the above result for Fock-adapted cocycles given in Lemma 1.2 of [LW05a].

(ii) The result extends immediately to right cocycles by use of the time-reversal operators ρ_t — see Lemma 1.3 below.

Let $\mathbb{S} = \mathrm{Lin}\{d\mathbf{1}_{[0,t[} : d \in \mathsf{k}, t \geqslant 0\}$, the subspace of $L^2(\mathbb{R}_+; \mathsf{k})$ consisting of right continuous, piecewise constant functions. It is a dense subspace, so $\mathcal{E} := \mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{S}\}$ is dense in $\mathcal{F}_+$. Consequently bounded operators on $\mathfrak{h} \otimes \mathcal{F}_+$ are determined by their inner products against vectors of the form $u\varepsilon(f)$ for $u \in \mathfrak{h}$, $f \in \mathbb{S}$.

The next result (essentially Proposition 6.2 of [LW00b]) follows immediately from adaptedness and (1.1).

Theorem 1.2. *Let X be a bounded adapted process. The following are equivalent*:

(i) *X is a left cocycle.*

(ii) *For each pair $c, d \in \mathsf{k}$, $(Q_t^{c,d} := E^{\varepsilon(c\mathbf{1}_{[0,t[})} X_t E_{\varepsilon(d\mathbf{1}_{[0,t[})})_{t\geqslant 0}$ is a semigroup on $\mathfrak{h}$, and for all $f, g \in \mathbb{S}$*

$$E^{\varepsilon(f\mathbf{1}_{[0,t[})} X_t E_{\varepsilon(g\mathbf{1}_{[0,t[})} = Q_{t_1-t_0}^{f(t_0),g(t_0)} \cdots Q_{t-t_n}^{f(t_n),g(t_n)} \tag{1.5}$$

where $\{0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant t\}$ contains the discontinuities of $f\mathbf{1}_{[0,t[}$ and $g\mathbf{1}_{[0,t[}$.

If in (i) *we replace left by right then* (1.5) *in* (ii) *must be replaced by*

$$E^{\varepsilon(f\mathbf{1}_{[0,t[})} X_t E_{\varepsilon(g\mathbf{1}_{[0,t[})} = Q_{t-t_n}^{f(t_n),g(t_n)} \cdots Q_{t_1-t_0}^{f(t_0),g(t_0)}. \tag{1.5$'$}$$

The collection of semigroups $\{Q^{c,d} : c, d \in \mathsf{k}\}$ is the family of *associated semigroups* of the cocycle X. Since the map $(c, d) \mapsto Q_t^{c,d}$ is jointly continuous, a cocycle X is determined by the operators $Q_t^{c,d}$ for c and d taken from a dense subset of k. In fact this observation can be further refined by using totality results such as those contained in [PS98, Ske00, Lin05] to show that it is sufficient to take c and d from a *total* subset of k that contains 0. If X is a C_0-cocycle then it is clear that all of the associated semigroups are strongly continuous, since the map $x \mapsto E_x$ is isometric. Conversely if all (or, rather, sufficiently many) of the associated semigroups are strongly continuous and if X is locally uniformly bounded, then from (1.5) it follows

that $t \mapsto X_t$ is weakly continuous at 0, and hence X is a C_0-cocycle. The a priori assumption of local boundedness is needed here for this (perhaps naive) method of proof to get weak continuity on all of the complete space $\mathfrak{h} \otimes \mathcal{F}_+$, rather than just $\mathfrak{h} \underline{\otimes} \mathcal{E}$, then Proposition 1.1 can be invoked to obtain the improved bound (1.4).

A stronger hypothesis on the map $t \mapsto X_t$ is *Markov-regularity*, as considered in [LW00b], which insists on norm continuity of the *Markov semigroup* $Q^{0,0}$. For a C_0-cocycle (or, indeed, any locally uniformly bounded cocycle) this is equivalent to assuming that *all* of the associated semigroups are norm continuous, which follows easily from the estimate $\|\varepsilon(a\mathbf{1}_{[0,t[}) - \varepsilon(c\mathbf{1}_{[0,t[})\| = O(t^{1/2})$.

In many cases, for instance when proving Theorem 2.1, it can be useful to pass from left to right cocycles or vice versa. Two methods for doing this are taking adjoints and time-reversal, where for any process X we define $\widetilde{X}$ by

$$\widetilde{X}_t := \rho_t(X_t)$$

with ρ_t is defined in (1.2).

Lemma 1.3. *Let X be a bounded adapted process. The following are equivalent*:

(i) *X is a left cocycle.*
(ii) *$X^* := (X_t^*)_{t \geqslant 0}$ is a right cocycle.*
(iii) *$\widetilde{X}$ is a right cocycle.*

Proof. Equivalence of (i) and (ii) is immediate since σ_t is *-homomorphic. Equivalence of (i) and (iii) follows from Theorem 1.2 and the fact that $R_t\varepsilon(c\mathbf{1}_{[0,t[}) = \varepsilon(c\mathbf{1}_{[0,t[})$, so that X and $\widetilde{X}$ share the same family of associated semigroups. □

Given a cocycle X we define two unital subalgebras of $B(\mathfrak{h})$:

$$\mathcal{A}_X = \text{the norm-closed algebra generated by } \{Q_t^{c,d} : c, d \in \mathsf{k}, t \geqslant 0\}, \tag{1.6}$$

and

$$\mathcal{M}_X = \text{the von Neumann algebra generated by } \mathcal{A}_X. \tag{1.7}$$

These algebras enter into characterisations of various properties of X. In the language of [LW05b] it follows from (1.5) (or (1.5)$'$) that $X_t \in \mathrm{M}(\mathcal{F}_+; \mathcal{A}_X)_\mathrm{b} \subset \mathrm{M}(\mathcal{F}_+; \mathcal{M}_X)_\mathrm{b} = \mathcal{M}_X \otimes B(\mathcal{F}_+)$, where $\mathrm{M}(\mathcal{F}_+; \mathsf{V})_\mathrm{b}$ denotes the $\mathcal{F}_+$-matrix space over an operator space V.

Proposition 1.4. *Let X be a left cocycle. We have the following sets of equivalences*:

(a) (i) *X is also a right cocycle.*
(ii) *$X = \widetilde{X}$.*
(iii) *$\mathcal{A}_X$ is commutative.*

(b) (i) *$X^* = \widetilde{X}$.*
(ii) *$(Q_t^{c,d})^* = Q_t^{d,c}$ for all $c, d \in \mathsf{k}$ and $t \geqslant 0$.*
In this case the algebra $\mathcal{A}_X$ is closed under taking adjoints.

(c) (i) *X is a self-adjoint cocycle.*
(ii) *$(Q_t^{c,d})^* = Q_t^{d,c}$ for all $c, d \in \mathsf{k}$ and $t \geqslant 0$, and $\mathcal{M}_X$ is commutative.*
In this case $X = \widetilde{X}$ as well.

Proof. (a) This is immediate from Theorem 1.2 and Lemma 1.3 since X is also a right cocycle if and only if not only (1.5) but also (1.5)$'$ holds.

(b) X is adapted and $\{\varepsilon(f) : f \in \mathbb{S}\}$ is total in $\mathcal{F}_+$, so $X^* = \widetilde{X}$ if and only if

$$\big(E^{\varepsilon(g\mathbf{1}_{[0,t[})} X_t E_{\varepsilon(f\mathbf{1}_{[0,t[})}\big)^* = E^{\varepsilon(f\mathbf{1}_{[0,t[})} X_t^* E_{\varepsilon(g\mathbf{1}_{[0,t[})} = E^{\varepsilon(f\mathbf{1}_{[0,t[})} \widetilde{X}_t E_{\varepsilon(g\mathbf{1}_{[0,t[})}$$

for all $f, g \in \mathbb{S}$ and $t \geqslant 0$. The result thus follows from Theorem 1.2 and Lemma 1.3, since $R_t \varepsilon(c\mathbf{1}_{[0,t[}) = \varepsilon(c\mathbf{1}_{[0,t[})$.

(c i $\Rightarrow$c ii) If X is self-adjoint then it is also a right cocycle by Lemma 1.3, thus $\mathcal{A}_X$ is commutative by (a iii), and so $X = X^* = \widetilde{X}$ by (a ii), so part (b) applies, which in particular shows that $\mathcal{M}_X$ is commutative.

(c ii $\Rightarrow$c i) Commutativity of $\mathcal{M}_X$ implies commutativity of $\mathcal{A}_X$, hence from (a) we have $X = \widetilde{X}$, and by (b) we have $X^* = \widetilde{X}$. □

Remarks. Commutativity of $\mathcal{A}_X$ does not imply commutativity of $\mathcal{M}_X$. To see this take any $A \in B(\mathfrak{h})$ then $(X_t = e^{tA} \otimes I_{\mathcal{F}_+})_{t \geqslant 0}$ is both a left and right cocycle, since $\sigma_r(X_t) = X_t$ for all $r, t \geqslant 0$. Moreoever $Q_t^{c,d} = e^{t(A + \langle c,d \rangle)}$, so that $\mathcal{A}_X$ is the unital algebra generated by A, which is certainly commutative, whereas $\mathcal{M}_X$ is the von Neumann algebra generated by A, so commutative if and only if A is normal.

This result also illustrates some relations that exist between the algebras defined through (1.6) and (1.7): for any cocycle X we have

$$\mathcal{A}_X = \mathcal{A}_{\widetilde{X}}, \quad \text{and} \quad \mathcal{M}_X = \mathcal{M}_{\widetilde{X}} = \mathcal{M}_{X^*},$$

but $\mathcal{A}_X$ need not equal $\mathcal{A}_{X^*}$. Other remarks on the differences between parts (a), (b) and (c) of the proposition are best made with reference to the stochastic generator of the cocycle, the subject of the next section, and so are postponed until then.

2. Generated cocycles

A major source of operator cocycles on Fock space comes from solutions of the left and right Hudson-Parthasarathy QSDEs:

$$dX_t = X_t F \, d\Lambda_t, \qquad X_0 = I, \tag{L}$$

$$dX_t = F X_t \, d\Lambda_t, \qquad X_0 = I. \tag{R}$$

Here the coefficient F is a bounded operator on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$, where the use of hats is defined by

$$\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}, \qquad \widehat{d} = \begin{pmatrix} 1 \\ d \end{pmatrix} \text{ for } d \in \mathsf{k}. \tag{2.1}$$

Moreover, let $P_{\mathsf{k}} \in B(\widehat{\mathsf{k}})$ denote the projection $\widehat{\mathsf{k}} \to \mathsf{k}$, and $\Delta := I_{\mathfrak{h}} \otimes P_{\mathsf{k}}$. Since $\mathfrak{h} \otimes \widehat{\mathsf{k}} \cong \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k})$, any $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ can and will be written as

$$F = \begin{bmatrix} A & B \\ C & D - I_{\mathfrak{h} \otimes \mathsf{k}} \end{bmatrix}$$

for $A \in B(\mathfrak{h})$, $B, C^* \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$ and $D \in B(\mathfrak{h} \otimes \mathsf{k})$.

Straightforward Picard iteration arguments ([Lin05]) produce solutions $X^F = (X_t^F)_{t \geqslant 0}$ of the left equation (L) and ${}^F X = ({}^F X_t)_{t \geqslant 0}$ of the right equation (R), although neither need be composed of bounded operators. However the solutions have domain $\mathfrak{h} \underline{\otimes} \mathcal{E}$ and satisfy a property called *weak regularity*, a property shared by any locally bounded process. Moreover X^F and ${}^F X$ are the unique weakly regular (weak) solutions to (L) and (R) for the given F. On the other hand, any weakly regular process X satisfies (L) (or (R)) (weakly) for at most one F. See Theorems 3.1 and 7.13 of [LW00a]. It follows that $(X^F)^*$ is the unique weakly regular solution of (R) for F^*, i.e. $(X^F)^*|_{\mathfrak{h} \underline{\otimes} \mathcal{E}} = {}^{F^*} X$.

The solution X^F enjoys a semigroup decompositions of the form (1.5), where now

$$Q_t^{c,d} = e^{t Z_d^c} \text{ for } Z_d^c = E^{\widehat{c}} (F + \Delta) E_{\widehat{d}} = E^{\widehat{c}} F E_{\widehat{d}} + \langle c, d \rangle. \tag{2.2}$$

The solution ${}^F X$ of (R) has a similar description involving the same semigroups, but with the product as in $(1.5)'$. One consequence is that a weakly regular process

X solves (L) if and only if the time-reversed process $\widetilde{X}$ satisfies (R). More importantly, *if* the solution to (L) (respectively to (R)) is a bounded process then it is a Markov-regular left (resp. right) cocycle. However it is still an open problem to determine all the operators F that yield bounded solutions. For contractive, isometric, coisometric, and hence unitary solutions the situation is understood much better, with the answer being given in terms of the map χ on $B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ where

$$\chi(F) := F + F^* + F^* \Delta F. \tag{2.3}$$

Theorem 2.1 ([Fag93, LW00a]). *Let $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$. We have the following sets of equivalences*:

(a) (i) ${}^F X$ *is contractive* (ii) X^F *is contractive* (iii) $\chi(F) \leqslant 0$ (iv) $\chi(F^*) \leqslant 0$

(b) (i) ${}^F X$ *is isometric* (ii) X^F *is isometric* (iii) $\chi(F) = 0$

(c) (i) ${}^F X$ *is coisometric* (ii) X^F *is coisometric* (iii) $\chi(F^*) = 0$

If, instead, we start with a Markov-regular C_0-cocycle X then all of its associated semigroups $Q^{c,d}$ are norm continuous and so have bounded generators Z^c_d. Let I be a set not containing 0, set $\widehat{I} = I \cup \{0\}$, and let $\{e_\alpha\}_{\alpha \in \widehat{I}}$ be an orthonormal basis of $\widehat{\mathsf{k}}$ with $e_0 = \binom{1}{0}$, so that $\{e_i\}_{i \in I}$ is an orthonormal basis of $0 \oplus \mathsf{k} \cong \mathsf{k}$. This basis induces the second of the following isomorphisms:

$$\mathfrak{h} \otimes \widehat{\mathsf{k}} \cong \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}) \cong \mathfrak{h} \oplus \bigoplus\nolimits^{(\dim \mathsf{k})} \mathfrak{h}. \tag{2.4}$$

Now define operators $\{F^\alpha_\beta : \alpha, \beta \in \widehat{I}\} \subset B(\mathfrak{h})$ through

$$\begin{aligned} &F^0_0 = Z^{e_0}_{e_0}, \quad F^i_0 = Z^{e_i}_{e_0} - Z^{e_0}_{e_0}, \quad F^0_j = Z^{e_0}_{e_j} - Z^{e_0}_{e_0} \quad \text{and} \\ &F^i_j = Z^{e_i}_{e_j} - Z^{e_i}_{e_0} - Z^{e_0}_{e_j} + Z^{e_0}_{e_0} - \delta^i_j I_{\mathfrak{h}}, \end{aligned} \tag{2.5}$$

for $i, j \in I$, and where δ^i_j is the Kronecker delta. If k is finite-dimensional, these F^α_β can be regarded as the components of the matrix associated to a bounded operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ through (2.4), and it follows that $X = X^F$ or ${}^F X$ as appropriate. That is, the cocycle is the solution of the relevant QSDE. For infinite-dimensional k, a priori the matrix $[F^\alpha_\beta]$ only gives us a form on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$, with respect to which X satisfies a weak form of (L) or (R) — this is Theorem 6.6 of [LW00b]. However, if the cocycle is in addition *contractive* then the form is bounded, and so the F^α_β are the components of some $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ as before.

Recall the subalgebras $\mathcal{A}_X$ and $\mathcal{M}_X$ of $B(\mathfrak{h})$ associated to a cocycle X by (1.6) and (1.7). For a generated cocycle X, i.e. one satisfying (L) or (R) for some $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, it follows from (2.2) and (2.5) that $\mathcal{A}_X$ is the unital algebra generated by the components F^α_β of F. That is, $F \in \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A}_X)_{\mathrm{b}}$, the $\widehat{\mathsf{k}}$-matrix space over $\mathcal{A}_X$. Moreover, from (2.2) and (2.5) we have

$$(Q^{c,d}_t)^* = Q^{d,c}_t \ \text{ for all } c, d \in \mathsf{k}, t \geqslant 0 \ \Leftrightarrow \ F = F^*.$$

The following is thus the infinitesimal version of Proposition 1.4:

Proposition 2.2. *Let $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ and suppose that X^F is bounded with locally uniform bounds, hence a Markov-regular left C_0-cocycle. We have the following sets of equivalences*:

(a) (i) $X^F = \widetilde{X^F} = {}^F X$.
 (ii) $F \in \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{C})_{\mathrm{b}}$ *for some commutative subalgebra* $\mathcal{C} \subset B(\mathfrak{h})$.

(b) (i) $\widetilde{X^F} = (X^F)^*$.
 (ii) $F = F^*$.

In this case $\mathcal{A}_X$ is closed under taking adjoints.

(c) (i) X^F *is self-adjoint.*
(ii) $F = F^*$ *and* $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ *for some commutative von Neumann algebra* $\mathcal{N}$.

Examples. (i) If $\mathfrak{h} = \mathsf{k} = \mathbb{C}$ and $F = \left[\begin{smallmatrix} -1/2 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ then $\mathcal{A}_X = \mathcal{M}_X = \mathbb{C}$, so $X^F = \widetilde{X^F} = {}^F X$, i.e. X^F is both a left and right cocycle. However $F \neq F^*$, and so (a) neither implies (b) nor (c). Furthermore, $\mathcal{A}_X$ being closed under taking adjoints does not imply $F = F^*$. In this example $X^F_t = W(\mathbf{1}_{[0,t[})$, the Weyl operator associated to $\mathbf{1}_{[0,t[} \in L^2(\mathbb{R}_+)$.

(ii) As noted after Proposition 1.4, commutativity of $\mathcal{A}_X$ does not imply commutativity of $\mathcal{M}_X$. This also shows that (a) does not imply (c).

(iii) Let $F = \left[\begin{smallmatrix} A & B \\ C & D-I \end{smallmatrix}\right]$ with $A = A^* = -\frac{1}{2}C^*C$, $B = C^*$, $D = D^*$, $D^2 = I$ and $(I + D)C = 0$. Then $F = F^*$ and $\chi(F) = 0$, so from Theorem 2.1 we have that X^F is unitary, and from (b) of the above proposition that $(X^F)^* = \widetilde{X^F}$. However if we ensure that $\mathcal{A}_X$ is not commutative then $(X^F)^* \neq X^F$; this can be achieved by taking $\mathfrak{h} = \mathbb{C}^2$, $\mathsf{k} = \mathbb{C}$ and $C = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. This shows that (b) implies neither (a) nor (c).

In the examples above it is the algebra generated by the components of F rather than the von Neumann algebra $\mathcal{N}_F$ generated by F itself that is of interest. For example in (i) $\mathcal{M}_X$ is commutative, whereas $\mathcal{N}_F$ is not, and the opposite holds true in the example in (iii).

Theorem 2.3. *Let* $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ *and suppose that* X^F *is bounded with locally uniform bounds, hence a Markov-regular left* C_0*-cocycle. The following are equivalent*:
(i) $X^F_t \geqslant 0$ *for all* $t \geqslant 0$.
(ii) $F = F^* \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ *for some commutative von Neumann algebra* $\mathcal{N}$, *and* $\Delta F \Delta + \Delta \geqslant 0$.

Proof. Given any von Neumann algebra $\mathcal{N}$ and $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$, if we define $\theta : \mathcal{N} \to \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ by $\theta(a) = F(a \otimes I_{\widehat{\mathsf{k}}})$ and assume that X^F is a bounded solution of (L) then the mapping process $k_t(a) := X^F_t(a \otimes I_{\mathcal{F}_+})$ is a solution of the Evans-Hudson QSDE $dk_t = k_t \circ \theta \, d\Lambda_t$. Moreover, by Theorem 4.1 of [LW00a], k is completely positive if and only if

$$\theta(a) = \psi(a) + E_{\widehat{0}}\, aK + K^* a E^{\widehat{0}} - a \otimes P_{\mathsf{k}} \tag{2.6}$$

for some completely positive map $\psi : \mathcal{N} \to \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ and $K \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}; \mathfrak{h})$.

(i $\Rightarrow$ ii) Suppose that each X^F_t is positive, then $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$ by Proposition 2.2, hence $(X^F_t)^{1/2} \in \mathcal{N} \otimes B(\mathcal{F}_+)$, and so commutes with $a \otimes I_{\mathcal{F}_+}$, showing that the flow k is completely positive. In particular, since $\Delta E_{\widehat{0}} = 0$,

$$\Delta F \Delta + \Delta = \Delta \theta(1) \Delta + \Delta = \Delta \psi(1) \Delta \geqslant 0.$$

(ii $\Rightarrow$ i) Write $F = \left[\begin{smallmatrix} A & B \\ B^* & D-I \end{smallmatrix}\right]$ so that $D \geqslant 0$. Since $\mathcal{N}$ is commutative,

$$\psi(a) := \begin{bmatrix} 0 & 0 \\ 0 & D(a \otimes I_{\mathsf{k}}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D^{1/2}(a \otimes I_{\mathsf{k}}) D^{1/2} \end{bmatrix}$$

is completely positive. Setting $K = \left[\tfrac{1}{2}A \quad B\right]$ we get

$$\psi(a) + E_{\widehat{0}}\, aK + K^* a E^{\widehat{0}} - a \otimes P_{\mathsf{k}} = F(a \otimes I_{\widehat{\mathsf{k}}}),$$

so θ has the form (2.6), and thus generates a completely positive flow. In particular $X^F_t = k_t(1)$ must be positive. $\square$

Corollary 2.4. *Let $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$. The following are equivalent*:

(i) *X^F is a positive contraction cocycle.*

(ii) *$F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$, $F \leqslant 0$ and $\Delta F \Delta + \Delta \geqslant 0$.*

(iii) *$F = \left[\begin{smallmatrix} A & B \\ B^* & D-I \end{smallmatrix}\right] \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$, with $A \leqslant 0$, $0 \leqslant D \leqslant I$ and $B = (-A)^{1/2} V (I-D)^{1/2}$ for some contraction $V \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$.*

Proof. If the flow k generated by $\theta(a) = F(a \otimes I_{\widehat{\mathsf{k}}})$ is completely positive then it is contractive if and only if $\theta(1) = F \leqslant 0$ ([LP98, Theorem 5.1] or [LW00a, Proposition 5.1]). Moreover if k is positive then $\|k_t\| = \|k_t(1)\| = \|X_t\|$. This gives the equivalence of (i) and (ii). Part (iii) follows from a standard characterisation of positive 2×2 operator matrices (e.g. [GLSW03, Lemma 2.1]). $\square$

Remark. In terms of the operators in (iii) one may recognise Bhat's characterisation of positive contraction cocycles in the special case when $\mathfrak{h} = \mathbb{C}$ ([Bha01, Theorem 7.5]). His focus there was on *local cocycles*, that is cocycles which satisfy $X_t \in \sigma_t\big(B(\mathfrak{h} \otimes \mathcal{F}_+)\big)'$ for all $t \geqslant 0$. For the CCR flow

$$\sigma_t\big(B(\mathfrak{h} \otimes \mathcal{F}_+)\big) = B(\mathfrak{h}) \otimes I_{\mathcal{F}_{[0,t[}} \otimes B(\mathcal{F}_{[t,\infty[}),$$

so this assumption is stronger than mere adaptedness, and forces X_t to act trivially on $\mathfrak{h}$, equivalently we must have $\mathcal{A}_X = \mathbb{C}$, or $F \in I_{\mathfrak{h}} \otimes B(\widehat{\mathsf{k}})$. Hence one may restrict to the case $\mathfrak{h} = \mathbb{C}$ without loss of generality.

The final characterisations rely on being able to multiply cocycles together to produce new cocycles.

Lemma 2.5. *Let $F, G \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ and suppose that the solutions X^F and X^G to* (L) *for these coefficients are both bounded with locally uniform bounds. Assume also that*

$$(F \otimes I_{\mathcal{F}_+}) \widehat{X_t^G} = \widehat{X_t^G} (F \otimes I_{\mathcal{F}_+}) \quad \text{for all } t \geqslant 0 \tag{2.7}$$

where $\widehat{X_t^G} \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F}_+)$ denotes the result of ampliating X_t^G to $\mathfrak{h} \otimes \widehat{\mathsf{k}} \otimes \mathcal{F}_+$. In this case the product $X^F X^G$ is a bounded left C_0-cocycle with stochastic generator $F + G + F\Delta G$.

Proof. The adjoint process $\big((X_t^F)^*\big)_{t \geqslant 0}$ is a right cocycle with stochastic generator F^*, and so the quantum Itô formula gives

$$\begin{aligned}
&\langle u\varepsilon(f), (X_t^F X_t^G - I) v\varepsilon(g)\rangle = \\
&\quad \int_0^t \Big\{ \langle \widehat{X_s^F}^* u \widehat{f}(s)\varepsilon(f), \widehat{X_s^G}(G \otimes I_{\mathcal{F}_+}) v \widehat{g}(s)\varepsilon(g)\rangle \\
&\qquad + \langle (F^* \otimes I_{\mathcal{F}_+}) \widehat{X_s^F}^* u \widehat{f}(s)\varepsilon(f), \widehat{X_s^G} v \widehat{g}(s)\varepsilon(g)\rangle \\
&\qquad + \langle (F^* \otimes I_{\mathcal{F}_+}) \widehat{X_s^F}^* u \widehat{f}(s)\varepsilon(f), (\Delta \otimes I_{\mathcal{F}_+}) \widehat{X_s^G}(G \otimes I_{\mathcal{F}_+}) v \widehat{g}(s)\varepsilon(g)\rangle \Big\}\, ds.
\end{aligned}$$

The commutativity assumed in (2.7) shows that the weakly regular process $X^F X^G$ satisfies (L) for $F + G + F\Delta G$, and so is a cocycle with this generator. $\square$

Remark. If $F \in \mathcal{M} \otimes B(\widehat{\mathsf{k}})$ and $G \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ then a sufficient condition for (2.7) is $\mathcal{N} \subset \mathcal{M}'$, since $\widehat{X_t^G} \in \mathcal{N} \otimes I_{\widehat{\mathsf{k}}} \otimes B(\mathcal{F}_+)$. In particular this is true if $\mathcal{M}$ is commutative and $\mathcal{N} = \mathcal{M}$.

Proposition 2.6. *Let $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$. The following are equivalent*:

(i) *X^F is an orthogonal projection-valued cocycle.*

(ii) $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ *for some commutative von Neumann algebra* $\mathcal{N}$*, and* $F + F^*\Delta F = 0$.
(iii) $F = \begin{bmatrix} -BB^* & B \\ B^* & P-I \end{bmatrix} \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ *for some commutative von Neumann algebra* $\mathcal{N}$ *where* $P \in \mathcal{N} \otimes B(\mathsf{k})$ *is an orthogonal projection, and* $BP = 0$.

Proof. (i $\Rightarrow$ ii) Since X^F is self-adjoint, $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for a commutative von Neumann algebra $\mathcal{N}$, and $F = F^*$. It follows that $(F \otimes I_{\mathcal{F}_+})\widehat{X_t^F} = \widehat{X_t^F}(F \otimes I_{\mathcal{F}_+})$, hence $(X^F)^2$ is a cocycle with stochastic generator $2F + F\Delta F$ by Lemma 2.5. But we assumed that $X_t^F = (X_t^F)^2$, and since generators are unique we get

$$F = 2F + F\Delta F = 2F + F^*\Delta F$$

as required.

(ii $\Rightarrow$ i) From $F + F^*\Delta F = 0$ it follows that F is self-adjoint, and that $F \leqslant 0$. Thus part (a) of Theorem 2.1 and part (c) of Proposition 2.2 apply to show that X^F is a self-adjoint contraction C_0-cocycle. But now Lemma 2.5 applies to show that $(X^F)^2$ is also a cocycle, with generator $2F + F\Delta F = F$, and so by uniqueness of solutions to (L) we have that each X_t^F is an orthogonal projection.

(ii $\Leftrightarrow$ iii) Simple algebra. □

The final characterisation rests on equivalences between operator (in-)equalities involving $\chi(F)$ defined in (2.3) and the following additional functions of F:

$$\pi(F) := F + F^* + F^*\Delta F + F\Delta F + F\Delta F^* + F\Delta F^*\Delta F \tag{2.8}$$
$$\varphi(F) := \chi(F) + \chi(F)\Delta\chi(F).$$

Lemma 2.7. *For any* $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ *we have the following sets of equivalences*:

(a) (i) $\chi(F) \leqslant 0$ (ii) $\chi(F^*) \leqslant 0$ (iii) $\varphi(F) \leqslant 0$ (iv) $\varphi(F^*) \leqslant 0$
(b) (i) $\pi(F) = 0$ (ii) $\pi(F^*) = 0$ (iii) $\varphi(F) = 0$ (iv) $\varphi(F^*) = 0$

Proof. (a) Since $\chi(F) = \chi(F)^*$, if $\varphi(F) \leqslant 0$ then $\chi(F) \leqslant -\chi(F)\Delta\chi(F) \leqslant 0$. Thus (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). However note that

$$\varphi(F) = (I + F^*\Delta)\chi(F^*)(I + \Delta F), \tag{2.9}$$

from which it follows that (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv).

(b) Now $\pi(F^*) = \pi(F)^*$ so (i) $\Leftrightarrow$ (ii). Also $\pi(F) = \chi(F^*)(I + \Delta F)$, hence (i) $\Rightarrow$ (iii) by (2.9). Finally, if $\varphi(F) = 0$ then $\chi(F^*) \leqslant 0$ by part (a), and so

$$0 = -\varphi(F) = \big[\big(-\chi(F^*)\big)^{1/2}(I + \Delta F)\big]^*\big[\big(-\chi(F^*)\big)^{1/2}(I + \Delta F)\big]$$

giving (iii) $\Rightarrow$ (i). □

Proposition 2.8. *Let* $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ *for a commutative von Neumann algebra* $\mathcal{N}$*. The following are equivalent*:

(i) X^F *is a partial isometry-valued cocycle.*
(ii) $\pi(F) = 0$*, where* $\pi(F)$ *is defined in* (2.8) *above.*

Proof. If X^F is partial isometry-valued then since $\mathcal{N}$ is commutative, the cocycle $(X^F)^*X^F$ is projection-valued with generator $\chi(F) = F + F^* + F^*\Delta F$ by Lemma 2.5. Hence $\varphi(F) = \chi(F) + \chi(F)\Delta\chi(F) = 0$ by Proposition 2.6 and so $\pi(F) = 0$ by the lemma above.

Conversely, if $\pi(F) = 0$ then $\varphi(F) = 0$ by the lemma, hence $\chi(F) \leqslant 0$ and so X^F is a contraction cocycle by part (a) of Theorem 2.1. Again Lemma 2.5 can be invoked to show that $(X^F)^*X^F$ is a (bounded) cocycle with generator $\chi(F)$ which satisfies the conditions of Proposition 2.6 and hence is projection-valued, so that X^F is itself a partial isometry-valued cocycle. □

The condition $\pi(F) = 0$ is necessarily satisfied by the generator of *any* Markov-regular partial isometry-valued cocycle, as can be shown by standard independence of quantum stochastic integrators/differentiation at zero arguments. In particular if $F = \left[\begin{smallmatrix} 0 & 0 \\ 0 & D-I \end{smallmatrix}\right]$ then $\pi(F) = 0$ if and only if D is a partial isometry, but for such pure-gauge cocycles this condition is in general *not* sufficient to imply that X^F is partial isometry-valued, as can be seen by using the explicit solution of (L) given in [Lin05, Example 5.3]. For each $n \in \mathbb{N}$ and $1 \leqslant j \leqslant n$ define $D_j^{(n)} \in B(\mathfrak{h} \otimes \mathsf{k}^{\otimes n})$ by having D act on $\mathfrak{h}$ and the jth copy of k, and ampliating to the other copies of k. Set

$$D^{(n)} := D_1^{(n)} \cdots D_n^{(n)} \in B(\mathfrak{h} \otimes \mathsf{k}^{\otimes n}).$$

Proposition 2.9. *Let $D \in B(\mathfrak{h} \otimes \mathsf{k})$ be a contraction and set $F = \left[\begin{smallmatrix} 0 & 0 \\ 0 & D-I \end{smallmatrix}\right]$. The following are equivalent*:

(i) *X^F is a partial isometry-valued cocycle.*
(ii) *$D^{(n)}$ is a partial isometry for each $n \in \mathbb{N}$.*

Proof. The symmetric tensor product of n copies of $L^2([0, t[; \mathsf{k})$ can be naturally identified with $L^2(\Delta_t^n; \mathsf{k}^{\otimes n})$, where $\Delta_t^n = \{0 < t_1 < \cdots < t_n < t\} \subset (\mathbb{R}_+)^n$ (see [Sch93] or [Lin05] for details). It follows that

$$\mathfrak{h} \otimes \mathcal{F}_+ \cong \left(\bigoplus_{n=0}^{\infty} L^2(\Delta_t^n; \mathfrak{h} \otimes \mathsf{k}^{\otimes n}) \right) \otimes \mathcal{F}_{[t,\infty[}$$

and that under this identification the solution X^F of (L) has the explicit form

$$X_t^F = \left(\bigoplus_{n=0}^{\infty} I_{L^2(\Delta_t^n)} \otimes D^{(n)} \right) \otimes I_{\mathcal{F}_{[t,\infty[}},$$

(see [Lin05]). The result follows. □

Example. As a special case, if $\mathsf{k} = \mathbb{C}$ then $\mathfrak{h} \otimes \mathsf{k}^{\otimes n} \cong \mathfrak{h}$ and $D^{(n)} = D^n$, the usual nth power of D. Thus in this setting X^F is partial isometry-valued if and only if D^n is a partial isometry for each n, whereas $\pi(F) = 0$ merely if D alone is a partial isometry. If we take $\mathfrak{h} = \mathbb{C}^2$ and the partial isometry

$$D = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}, \qquad \theta \notin \pi\mathbb{Z}/2,$$

then $D^2(D^2)^* D^2 \neq D^2$, so that X^F is *not* partial isometry-valued.

An operator D such that D^n is a partial isometry for each $n \in \mathbb{N}$ is called *power partial isometry*; these have been characterised by Halmos and Wallen ([HW70]).

3. A transformation of positive cocycles

Proposition 3.1. *Let $X = (X_t)_{t \geqslant 0}$ be a left cocycle with $X_t \geqslant 0$ for each $t \geqslant 0$. Then for each real number $\alpha > 0$ the family $X^\alpha = (X_t^\alpha)_{t \geqslant 0}$ is a left cocycle.*

Proof. We are dealing with a self-adjoint cocycle, so it is both a left and a right cocycle, hence

$$X_{r+t} = X_r \sigma_r(X_t) = \sigma_r(X_t) X_r \quad \text{for all } r, t \geqslant 0. \tag{3.1}$$

From this it is clear that X^n is a cocycle for any integer $n \geqslant 1$, and one consisting of positive operators.

Since σ_r is a *-homomorphism it follows that $\sigma_r(X_t) \geqslant 0$ and that $\sigma_r(X_t^{1/2}) = \sigma_r(X_t)^{1/2}$. Moreover from (3.1) and the continuous functional calculus we obtain

$$[X_r^{1/2}, \sigma_r(X_t)^{1/2}] = [X_r^{1/2}, \sigma_r(X_t^{1/2})] = 0,$$

and thus
$$X_r^{1/2}\sigma_r(X_t^{1/2}) \geqslant 0, \qquad \big(X_r^{1/2}\sigma_r(X_t^{1/2})\big)^2 = X_{r+t}.$$
Hence, by uniqueness of positive square roots, $X^{1/2}$ is a left and right cocycle of positive operators.

These two observations show that X^α is a cocycle for any dyadic rational α. To get the desired result for any $\alpha > 0$ let $(\alpha_n)_{n\geqslant 1}$ be a decreasing sequence of dyadic rationals with $\alpha_n \to \alpha$. Now if $h_\beta(t) := t^\beta$ for $\beta > 0$ then $h_{\alpha_n} \to h_\alpha$ locally uniformly — the function sequence is pointwise increasing on $[0,1]$ and pointwise decreasing on $[1,T]$ for any $T > 1$, and so Dini's Theorem may be applied. Thus appealing to the continuous functional calculus once more and continuity of σ_r is enough to show that $X^\alpha_{r+t} = X^\alpha_r \sigma_r(X^\alpha_t)$ as required. □

It should be noted that the proof of the above result does not on particular properties of the CCR flow σ on Fock space. Indeed, the result is valid for any E-semigroup since even preservation of the identity by σ is not used.

However, if the cocycle X is a Markov-regular positive contraction cocycle then it has a stochastic generator F. The next results discuss how F is transformed by taking powers of X, and this is mediated through the following functions from the algebra $C[0,1]$, defined for each $\alpha > 0$.
$$f_\alpha(t) = \begin{cases} \dfrac{\alpha - 1 - \alpha t + t^\alpha}{(1-t)^2} & \text{if } t < 1, \\ \frac12\alpha(\alpha-1) & \text{if } t = 1, \end{cases} \qquad g_\alpha(t) = \begin{cases} \dfrac{1-t^\alpha}{1-t} & \text{if } t < 1, \\ \alpha & \text{if } t = 1, \end{cases} \qquad h_\alpha(t) = t^\alpha.$$
Note that h_α is a homeomorphism $[0,1] \to [0,1]$, so induces an automorphism of $C[0,1]$ by composition. Also we have the following identities, valid for all $\alpha, \beta > 0$:
$$g_\alpha = \alpha - (1-h_1) f_\alpha; \tag{3.2a}$$
$$f_\alpha + f_\beta + g_\alpha g_\beta = f_{\alpha+\beta}, \quad g_\beta + g_\alpha h_\beta = g_{\alpha+\beta}, \quad h_\alpha h_\beta = h_{\alpha+\beta}; \tag{3.2b}$$
$$\beta f_\alpha + g_\alpha^2 (f_\beta \circ h_\alpha) = f_{\alpha\beta}, \quad g_\alpha (g_\beta \circ h_\alpha) = g_{\alpha\beta}, \quad h_\beta \circ h_\alpha = h_{\alpha\beta}; \tag{3.2c}$$
and the inequalities
$$f_\alpha(t) \leqslant f_\beta(t), \quad g_\alpha(t) \leqslant g_\beta(t), \quad h_\alpha(t) \geqslant h_\beta(t), \tag{3.2d}$$
valid for all $t \in [0,1]$ and $1 \leqslant \alpha < \beta$.

Lemma 3.2. *Suppose $F = \left[\begin{smallmatrix} A & B \\ B^* & D-I \end{smallmatrix}\right] \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ is the generator of a positive contraction cocycle. Then so is $F_\alpha \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ where*
$$F_\alpha := \begin{bmatrix} \alpha A + B f_\alpha(D) B^* & B g_\alpha(D) \\ g_\alpha(D) B^* & h_\alpha(D) - I \end{bmatrix}. \tag{3.3}$$

Remark. The proof uses the following elementary fact: if K_1, K_2 and K_3 are Hilbert spaces, and $S \in B(\mathsf{K}_1; \mathsf{K}_2)$, $T \in B(\mathsf{K}_1; \mathsf{K}_3)$ such that $S^*S \leqslant T^*T$, then there is a contraction $W \in B(\mathsf{K}_2; \mathsf{K}_3)$ such that $S = WT$. This follows since the inequality allows us to define W by setting $W(T\xi) = S\xi$ on $\operatorname{Ran} T$ and $W|_{(\operatorname{Ran} T)^\perp} = 0$.

Proof. By condition (iii) of Corollary 2.4, $F \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for some commutative von Neumann algebra $\mathcal{N}$, $A \leqslant 0$, $0 \leqslant D \leqslant I$ and there is a contraction V such that $B = (-A)^{1/2} V (I-D)^{1/2}$.

Now $F_\alpha \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$, and $0 \leqslant h_\alpha(D) \leqslant I$ since $h_\alpha([0,1]) = [0,1]$. Also $g_\alpha \geqslant 0$, so
$$\begin{aligned} 0 &\leqslant (-A)^{1/2} V g_\alpha(D) V^* (-A)^{1/2} \\ &= (-A)^{1/2} V (\alpha - (I-D)^{1/2} f_\alpha(D) (I-D)^{1/2}) V^* (-A)^{1/2} \\ &\leqslant -\alpha A - B f_\alpha(D) B^*, \end{aligned}$$

using (3.2a). These inequalities prove the existence of a contraction $W \in B(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ that satisfies

$$W(-\alpha A - Bf_\alpha(D)B^*)^{1/2} = g_\alpha(D)^{1/2} V^* (-A)^{1/2},$$

and so

$$\begin{aligned} Bg_\alpha(D) &= (-A)^{1/2} V g_\alpha(D)^{1/2} (I - D)^{1/2} g_\alpha(D)^{1/2} \\ &= (-\alpha A - Bf_\alpha(D)B^*)^{1/2} W^* (I - h_\alpha(D))^{1/2}, \end{aligned}$$

since $(I - D)g_\alpha(D) = I - h_\alpha(D)$. Thus F_α satisfies condition (iii) of Corollary 2.4, showing that it is the generator of a positive contraction cocycle. □

Theorem 3.3. *Let X be a Markov-regular positive contraction cocycle with stochastic generator F. Then for each real $\alpha > 0$ the cocycle X^α is Markov-regular with generator F_α given by* (3.3).

Proof. The identities (3.2b) lead immediately to

$$F_\alpha + F_\beta + F_\alpha \Delta F_\beta = F_{\alpha+\beta} \quad \text{for all } \alpha, \beta > 0. \tag{3.4}$$

In particular, noting that $F_1 = F$, it follows from Lemma 2.5 and an induction argument that X^n is Markov-regular and has generator F_n for each $n \in \mathbb{N}$.

Next, Lemma 3.2 shows that $F_{\frac{1}{2}}$ is the generator of some positive contraction cocycle Y. By (3.4) and Lemma 2.5, Y^2 is also a cocycle with generator $2F_{\frac{1}{2}} + F_{\frac{1}{2}} \Delta F_{\frac{1}{2}} = F_1 = F$, and so $Y^2 = X$ by uniqueness of generators. Thus the cocycle $X^{1/2}$ has generator $F_{\frac{1}{2}}$.

Now the identities (3.2c) give $(F_\alpha)_\beta = F_{\alpha\beta}$ for all $\alpha, \beta > 0$, so square roots may be taken repeatedly, followed by taking arbitrarily large integer powers to show that X^α is Markov-regular with generator F_α for each dyadic rational $\alpha > 0$.

If we choose any real number $\alpha > 1$ and let $(\alpha_n)_{n \geqslant 1}$ be a sequence of such rationals with $\alpha_n \downarrow \alpha$, then the function sequences (f_{α_n}), (g_{α_n}) and (h_{α_n}) converge pointwise to f_α, g_α and h_α. The inequalities (3.2d) show that the convergence is also monotonic, and hence uniform by Dini's Theorem, so that $F_{\alpha_n} \to F_\alpha$ in norm. It follows from (1.5) and (2.2) that the associated semigroups of the cocycle X^α are the norm limits of the semigroups associated to X^{α_n}, and thus X^α has stochastic generator F_α.

Finally, for any remaining $0 < \alpha < 1$ pick $n \in \mathbb{N}$ so that $\beta := 2^n \alpha > 1$, then X^β has generator F_β, and $X^\alpha = (X^\beta)^{2^{-n}}$ has generator $(F_\beta)_{2^{-n}} = F_\alpha$. □

4. Polar decomposition

One obvious question to ask given the results above is the following: if X is a contraction cocycle such that $\mathcal{M}_X$ is commutative then we can form the positive part process $(|X_t|)_{t \geqslant 0} = \big((X_t^* X_t)^{1/2}\big)_{t \geqslant 0}$ which is again a cocycle, so can we choose partial isometries U_t so that $U_t |X_t| = X_t$ for each t *and* so that $(U_t)_{t \geqslant 0}$ is a cocycle?

What follows answers this question when X is Markov-regular with generator $F = \left[\begin{smallmatrix} A & B \\ C & D-I \end{smallmatrix}\right]$. The necessary and sufficient conditions on F for contractivity of X is $\chi(F) \leqslant 0$ (Theorem 2.1) which translates as:

$$\begin{gathered} \|D\| \leqslant 1, \quad A + A^* + C^*C \leqslant 0, \quad \text{and} \\ B + C^*D = (-A - A^* - C^*C)^{1/2} V (I - D^*D)^{1/2} \end{gathered} \tag{4.1}$$

for some contraction $V \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h})$. Lemma 2.5, Proposition 3.1 and Theorem 3.3 combine to show that $(|X_t|)_{t \geqslant 0}$ is a Markov-regular cocycle with generator $G =$

$\chi(F)_{\frac{1}{2}}$, which equals

$$\begin{bmatrix} \frac{1}{2}(A + A^* + C^*C) + (B + C^*D)f_{\frac{1}{2}}(|D|^2)(B^* + D^*C) & (B + C^*D)g_{\frac{1}{2}}(|D|^2) \\ g_{\frac{1}{2}}(|D|^2)(B^* + D^*C) & |D| - I \end{bmatrix}$$

Now suppose that U is a partial isometry-valued cocycle with generator $E = \left[\begin{smallmatrix} K & L \\ M & N-I \end{smallmatrix}\right] \in \mathcal{M}_X \otimes B(\widehat{\mathsf{k}})$. Then the product process $(U_t|X_t|)_{t\geqslant 0}$ is a cocycle with generator $E + G + E\Delta G$ (Lemma 2.5). This must equal F to give $U|X| = X$, and thus K, L, M and N must be chosen to satisfy

$$N|D| = D, \tag{4.2a}$$

$$M = C - Ng_{\frac{1}{2}}(|D|^2)(B^* + D^*C), \tag{4.2b}$$

$$L|D| = B - (B + C^*D)g_{\frac{1}{2}}(|D|^2), \tag{4.2c}$$

$$\begin{aligned} K = \tfrac{1}{2}(A - A^* - C^*C) - (B + C^*D)f_{\frac{1}{2}}(|D|^2)(B^* + D^*C) \\ - Lg_{\frac{1}{2}}(|D|^2)(B^* + D^*C). \end{aligned} \tag{4.2d}$$

Note that N and L are fixed on $\operatorname{Ran}|D|$ by these equations, and once these are chosen the operators K and M are defined by (4.2d) and (4.2b) respectively. Also, we want U to be partial isometry-valued, so need to satisfy $\pi(E) = 0$ by Proposition 2.8. This is equivalent to requiring

$$N = NN^*N, \quad M^*N + LN^*N = 0 \ \text{ and } \ K + K^* + M^*M + L(I - N^*N)L^* = 0 \tag{4.3}$$

So now if we choose any partial isometry N that satisfies (4.2a) then one solution to this problem is obtained by setting

$$L = -C^*N + (B + C^*D)g_{\frac{1}{2}}(|D|^2).$$

Since $\sqrt{t}\, g_{\frac{1}{2}}(t) = 1 - g_{\frac{1}{2}}(t)$ it is easy to check (4.2c); moreover the second equation in (4.3) follows. Checking that the third equation holds is much more tedious, but is greatly assisted by noting that $2f_{\frac{1}{2}} + (g_{\frac{1}{2}})^2 = 0$. Summarising the above we have:

Theorem 4.1. *Every Markov-regular contraction cocycle X for which $\mathcal{M}_X$ is commutative can be written as the product of a partial isometry-valued cocycle and a positive contraction cocycle.*

Examples. (i) Take $\mathfrak{h} = \mathbb{C}$, then $\mathcal{A}_X = \mathcal{M}_X = \mathbb{C}$, thus Theorem 4.1 is applicable to *any* contraction cocycle on $\mathcal{F}_+$, since also every cocycle is trivially Markov-regular in this context. Now the generator of such a cocycle is $F \in B(\mathbb{C} \oplus \mathsf{k})$ which can be written as

$$F = \begin{bmatrix} i\mu - \frac{1}{2}(\nu^2 + \|v\|^2) & \langle \nu(I - D^*D)^{1/2}w - D^*v| \\ |v\rangle & D - I \end{bmatrix}$$

for some choice of $v \in \mathsf{k}$, $\nu \in [0, \infty[$, $\mu \in \mathbb{R}$, contraction $D \in B(\mathsf{k})$ and $w \in \mathsf{k}$ with $\|w\| \leqslant 1$. This follows from (4.1) (see also [Lin05, Theorem 5.12] or [GLSW03, Theorem 6.2]). It is then possible to write down the generators G and E of the positive part and partial isometry cocycles for any choice of partial isometry N such that $N|D| = D$.

As a particular case suppose that D is already a partial isometry, let $P = |D|$, the projection onto the initial space of D, and take $N = D$. Then

$$G = \begin{bmatrix} -\frac{\nu^2}{2}\left(1 + \|P^\perp w\|^2\right) & \langle \nu P^\perp w| \\ |\nu P^\perp w\rangle & P - I \end{bmatrix} \quad \text{and}$$

$$E = \begin{bmatrix} i\mu - \frac{1}{2}\|v\|^2 - \frac{1}{2}\nu^2\|P^\perp w\| & \langle \nu P^\perp w - D^*v| \\ |v\rangle & D - I \end{bmatrix}.$$

This applies for example if we take $\mathsf{k} = l^2(\mathbb{N})$ with usual orthonormal basis and take D to be the coisometric left shift, so that P is the projection onto $\{e_1\}^\perp$. Note that if we choose $\mu = \nu = 0$, and $v = w = 0$ then

$$F = E = \begin{bmatrix} 0 & 0 \\ 0 & D - I \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & P - I \end{bmatrix}.$$

If $\Gamma(Z)$ denotes the second quantisation of $Z \in B(L^2(\mathbb{R}_+; \mathsf{k}))$ then using the isomorphism $L^2(\mathbb{R}_+; \mathsf{k}) \cong L^2(\mathbb{R}_+) \otimes \mathsf{k}$ it follows that

$$X_t = U_t = \Gamma(M_{\mathbf{1}_{[0,t[}} \otimes D + M_{\mathbf{1}_{[t,\infty[}} \otimes I_{\mathsf{k}}), \quad |X_t| = \Gamma(M_{\mathbf{1}_{[0,t[}} \otimes P + M_{\mathbf{1}_{[t,\infty[}} \otimes I_{\mathsf{k}}),$$

where M_f denotes multiplication by $f \in L^\infty(\mathbb{R}_+)$. In particular since D is not normal, the algebra generated by the process X is not commutative.

On a different tack, note that equation (4.2c) only specifies L on $\operatorname{Ran}|D|$, so one might be tempted to set it equal to 0 on the orthogonal complement, which would certainly be the case if we replace L by $L' := LN^*N$. This has the effect of apparently making it easier to check the third of the identities in (4.3) (noting that the first and second remain valid), since $L'(I - N^*N)L'^* = 0$. But for our example above with D as the left shift one finds that K now becomes

$$K = i\mu - \tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\nu^2\|P^\perp w\|^2 \quad \Rightarrow \quad K + K^* + M^*M = \nu^2\|P^\perp w\|^2.$$

Thus the third equation in (4.3) will fail for an appropriate choice of ν and w, and hence U will not be partial isometry-valued.

(ii) As a special case of the more general situation, suppose that $F = \left[\begin{smallmatrix} A & B \\ C & D-I \end{smallmatrix}\right] \in \mathcal{N} \otimes B(\widehat{\mathsf{k}})$ for a commutative von Neumann algebra $\mathcal{N}$, and with D isometric. Then $|D| = I$, $N = D$ and hence

$$G = \begin{bmatrix} \frac{1}{2}(A + A^* + C^*C) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} \frac{1}{2}(A - A^* - C^*C) & -C^*D \\ C & D - I \end{bmatrix}.$$

In particular the positive part is $|X_t| = P_t \otimes I_{\mathcal{F}_+}$ where P_t is the positive semigroup on $\mathfrak{h}$ with generator $\frac{1}{2}(A + A^* + C^*C)$, and all of the stochastic terms occur only in the process U. Moreover in this case $\chi(E) = 0$, so U is an isometric cocycle.

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